



A RELATION BETWEEN THE FORMULATIONS OF THE BOUNDARY ELEMENT METHOD†

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A relation between the formulations of the boundary element method (BEM) based on boundary integral equations (BIEs) and the method of weighted errors (MWE) [1, 2] and the variational formulations [3–11] is established, which can be used not only to solve problems in the theory of elasticity but also other problems of mathematical physics. Copyright © 1996 Elsevier Science Ltd.

The formulation of the boundary element method (BEM) based on boundary integral equations (BIEs) is essentially a numerical method of solving these equations: the desired density at the points of the boundary element can be approximated by the interpolants of the finite element method (FEM), a BIE being written for each nodal point of the discrete boundary (the collocation method). Thus, the problem can be reduced to a system of discrete boundary equations (DBEs) for the nodal values of the desired density. The formulation of the BEM on the basis of the method of weighted errors (MWE) involves reducing the approximate solution of the boundary-value problem to solving the problem of minimizing the error in the boundary conditions at the points of the discrete boundary. In doing so the well-known integral relations based on Green's formulae are used. These connect the values of functions at the points of the domain and the boundary written for the approximation of the desired solution and the singular (fundamental) solution of the differential equation of the boundary-value problem by the FEM interpolants.

The variational formulations of the BEM, which are the gist of the variational method of boundary elements, make use of problems concerned with minimizing the boundary functionals (BF) or the generalized Trefftz functionals (GTF) of the original boundary-value problems for admissible functions of the form of discrete boundary potentials whose density can be approximated by the FEM interpolants. The relation between the variational formulation of the BEM and the formulation based on the MWE is obvious: the variational boundary equations obtained by minimizing the BF (or GTF) can be regarded as the relations of the MWE for the constructed boundary-element approximations "in Ritz's sense" of the solution of the problem and the given boundary values.

As in the case of weak variational formulations of the FEM, when "variational" equations of Galerkin type are used [12, 13], a weak variational formulation of the MBE is possible. To solve the Neumann boundary-value problem for Laplace's equation this formulation uses the direct interpolation of the normal derivative at the points of the discrete boundary and a discrete condition corresponding to the condition for the normal derivative of a harmonic function, which is satisfied a priori in the formulation "in Ritz's sense".

Below we shall establish (using the solution of the Saint-Venant problem as an example) that the DBEs obtained in the weak formulation are the same as the DBEs resulting from the formulation of the MBE based on the BIE, thereby establishing a relation between the formulations.

1. The Saint-Venant problem of the twisting of a homogeneous isotropic rod, stated as a problem in mathematical physics, corresponds [14] to the non-homogeneous Neumann boundary-value problem for Laplace's equation

$$\begin{aligned} \Delta\varphi(x) &= 0, \quad x \in G; \quad \partial_\nu\varphi|_S = f(y), \quad y \in S \\ f &= y^{(2)} \cos(\nu, x^{(1)}) - y^{(1)} \cos(\nu, x^{(2)}) \end{aligned} \quad (1.1)$$

Classical potential theory can be used to solve this problem. The desired solution can be represented as a harmonic simple-layer potential (SLP) with unknown density [15]

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$$\varphi(x) = \int_S \psi(y) \ln \frac{1}{r} ds(y), \quad x \in \bar{G}, \quad y \in S \quad (1.2)$$

Here G is the region of intersection of the rod and the boundary S and $r = |x - y|$. Note that φ exists provided that S is sufficiently smooth (a Lyapunov surface [15]) and ψ is a continuous function at the points of S . In this connection we observe that the algorithm of the BEM (the indirect formulation [1, 2]) makes use of an approximating SLP in the form of the sum of integrals over boundary elements, at the points of which the density interpolation is a continuous function.

In what follows we shall retain the notation of [3, 7]: let Δs_n be a linear boundary element and let $S_\Delta = \cup \Delta s_n$ ($n = 1, \dots, N$) be the discrete boundary of the polygonal cross-section G_Δ of the rod. In global (Cartesian) coordinates $y = (y^{(1)}, y^{(2)})$ the approximating SLP for (1.2) has the form

$$\varphi_N(x) = \int_{S_\Delta} \psi_\Delta(y) \ln \frac{1}{r_\Delta} ds_\Delta(y) = \sum_{n=1}^N \int_{\Delta s_n} \psi_n(y) \ln \frac{1}{r_n} ds(y_n), \quad x \in \bar{G}_\Delta \quad (1.3)$$

where $y_n^{(i)}(\eta)$ ($i = 1, 2$) is the relation, at the points of Δs_n , between the global coordinates and the local coordinate $\eta \in [-1, 1]$ of the linear element

$$y_n^{(i)}(\eta) = \sum_{k=1}^2 Y_{nk}^{(i)} \psi_k = \bar{Y}_n^{(i)} + \bar{Y}_n^{(i)} \eta \quad (1.4)$$

$$\bar{Y}_n^{(i)} = \frac{1}{2}(Y_{n1}^{(i)} + Y_{n2}^{(i)}), \quad \bar{Y}_n^{(i)} = \frac{1}{2}(Y_{n2}^{(i)} - Y_{n1}^{(i)})$$

Here $Y_{nk}^{(i)}$ ($i = 1, 2$) are the Cartesian coordinates of the nodes Δs_n .

In the general case the interpolation nodes (the functional nodes) may not coincide with the geometric nodes $k = 1, 2$. Then, given the interpolation $\psi_n = \sum \Psi_{nk'} \psi_{k'}(\eta)$, $k' = 1, \dots, K'$, where $\Psi_{nk'}$ are the desired nodal values and $\psi_{k'}$ are the basis functions of the BEM, the integral over Δs_n in (1.3) can be written as

$$\int_{-1}^{+1} \sum_{k'=1}^{K'} \Psi_{nk'} \psi_{k'} \ln \frac{1}{r_n} |J_n| d\eta \quad (1.5)$$

where $|J_n|$ is the Jacobian (the determinant of the Jacobi matrix) of the transformation $y_n(\eta)$

$$|J_n| = \left\{ \sum_{i=1}^2 (\partial_\eta y_n^{(i)})^2 \right\}^{1/2} = \left\{ \sum_{i=1}^2 (\bar{Y}_n^{(i)})^2 \right\}^{1/2} \quad (1.6)$$

We will consider the simplest case (which is sufficient, since the interpolations agree to within an order of magnitude, see below) of constant interpolation ψ_n , i.e. when the node k' lies between the nodes $k = 1$ and $k = 2$. Here $\psi_{k'} \equiv 1$ and the nodal value of the density are denoted by Ψ_{n0} ($n = 1, \dots, N$). In this case the discrete BIE at the points Δs_n can be written as

$$\pi \Psi_{n0} + \Psi_{n0} \int_{\Delta s_n} \partial_{v_n}(x) \left(\ln \frac{1}{r_n} \right) ds(y_n) = f_n(y_n) \quad (1.7)$$

where the first term on the left-hand side corresponds to a nodal jump of the normal derivative of the approximating SLP and f_n is a given value of the normal derivative at the points of Δs_n . The integral coefficient in (1.7) (as a contribution of Δs_n) can be computed in the global coordinates between the limits of integration from $Y_{n1}^{(i)}$ to $Y_{n2}^{(i)}$. To do so we use the equality

$$\frac{\partial}{\partial x^{(i)}} \ln r_n = \frac{x^{(i)} - y_n^{(i)}}{r_n^2}, \quad i = 1, 2$$

and the following substitution for the variable of integration

$$dr_n = \frac{\partial r_n}{\partial y_n^{(i)}} dy_n^{(i)} = -\frac{x^{(i)} - y_n^{(i)}}{r_n} dy_n^{(i)}, \quad i = 1, 2$$

Here $r_n = |x - y_n|$, $x \in S_\Delta$, $y_n \in \Delta s_n$ and the values of the direction cosines of the outer normal vector v_n at the points of Δs_n [7] are also used, see (1.4) and (1.6)

$$\cos \alpha_{1n} \equiv \cos(v_n, x^{(1)}) = \frac{\bar{Y}_n^{(2)}}{|J_n|}, \quad \cos \alpha_{2n} \equiv \cos(v_n, x^{(2)}) = -\frac{\bar{Y}_n^{(1)}}{|J_n|} \tag{1.8}$$

As a result, we obtain for any n

$$\int_{\Delta s_n} \partial_{v_n}(x) \left(\ln \frac{1}{r_n} \right) ds(y_n) = -(k_n - k_n^{-1}) \ln \frac{r_{n2x}}{r_{n1x}} \tag{1.9}$$

$$k_n = -\frac{\bar{Y}_n^{(2)}}{\bar{Y}_n^{(1)}}, \quad r_{nkx} = |x - Y_{nk}|, \quad x \in S_\Delta, \quad Y_{nk} \in \Delta s_n$$

It follows that when the DBEs are written for $x = Y_{m0}$, the contributions of the boundary element Δs_n depend on the distance from the functional node Y_{m0} of the constant interpolation density ψ_n to the geometric nodes Y_{nk} ($k = 1, 2; n = 1, \dots, N$). We use (1.4) and (1.6) to approximate the given value of the normal derivative of $f(y)$ at the points of Δs_n (see (1.1))

$$f_n = \frac{1}{|J_n|} \sum_{i=1}^2 Y_{n0}^{(i)} \bar{Y}_n^{(i)}, \quad \forall n \tag{1.10}$$

As a result, the system of DBEs based on the discrete BIE (1.7) for constant functional interpolation, taking (1.9) and (1.10) into account, can be written as

$$\sum_{n=1}^N \Psi_{n0} a_{nm} = \sum_{n=1}^N f_n, \quad m = 1, \dots, N \tag{1.11}$$

where

$$a_{nm} = \pi + c_n \ln \frac{r_{n2Y_{m0}}}{r_{n1Y_{m0}}} \quad (c_n = k_n^{-1} - k_n)$$

are the coefficients of the linear algebraic system of equations for the nodal values Ψ_{n0} of the density of the approximating SLP.

It is obvious that for fixed m each term $\Psi_{n0} a_{nm}$ on the left-hand side of (1.11) can be regarded as a normal derivative at Δs_n ($n = 1, \dots, N$), since this is so for each term on the right-hand side (this also follows from the discrete BIE (1.7)). The aforesaid serves as a justification of the following interpolation problem: let $\{\Psi_{n0}\}_{n=1, \dots, N}$ be a solution of system (1.11); it is required to determine the nodal values Φ_{nk} ($k = 1, 2$ being the geometric nodes of Δs_n) of the interpolation of some function $\tilde{\varphi}_n(y)$, $y \in \Delta s_n$ whose normal derivative at the points of Δs_n is equal to $\Psi_{n0} a_{nm}$ (the constant interpolation ψ_n and linear interpolation $\tilde{\varphi}_n$ being consistent to within an order of magnitude).

2. For the linear interpolation $\tilde{\varphi}_n(y_n(\eta)) = \sum \Phi_{nk} \psi_k$ ($k = 1, 2$), to compute the normal derivative $\partial_{v_n} \tilde{\varphi}_n$ one can use [3, 7] the function $\psi_k(y_n(\eta))$, differentiated as follows:

$$\frac{\partial \psi_k}{\partial y_n^{(i)}} = \frac{\partial \psi_k}{\partial \eta} \frac{\partial \eta}{\partial y_n^{(i)}}, \quad i = 1, 2; \quad k = 1, 2$$

Here $\partial \eta / \partial y_n^{(i)} = (\partial y_n^{(i)} / \partial \eta)^{-1}$, and the direction cosines are given by (1.8). As a result, we obtain

$$\partial_{v_n} \tilde{\Phi}_n = \frac{1}{|J_n|} \left(\frac{\partial_{\eta} y_n^{(2)}}{\partial_{\eta} y_n^{(1)}} - \frac{\partial_{\eta} y_n^{(1)}}{\partial_{\eta} y_n^{(2)}} \right) \sum_{k=1}^2 \Phi_{nk} \partial_{\eta} \Psi_k, \quad \left(\partial_{\eta} y_n^{(i)} = \frac{\partial y_n^{(i)}}{\partial \eta} \right)$$

Taking into account the relation between the global coordinates and the local coordinate (see (1.4)) as well as (1.11), we find that the expression in square brackets on the right-hand side of the last equality is equal to

$$\bar{Y}_n^{(2)} (\bar{Y}_n^{(1)})^{-1} - \bar{Y}_n^{(1)} (\bar{Y}_n^{(2)})^{-1} = k_n^{-1} - k_n = c_n, \quad \forall n$$

It follows that the interpolation of the normal derivative for the linear interpolation $\tilde{\Phi}_n$ at the points of Δs_n has the form

$$\partial_{v_n} \tilde{\Phi}_n = \frac{c_n}{|J_n|} \sum_{k=1}^2 \Phi_{nk} \partial_{\eta} \Psi_k, \quad \forall n \tag{2.1}$$

For the linear basis functions of the BEM (see (1.4)) it is easily verified that if the equality $\Phi_{n1} = \Phi_{n2} = \Phi_n$ holds for the nodal values (consequently, $\tilde{\Phi}_n = \Phi_n \Sigma \Psi_k = \Phi_n$ is constant at the points of Δs_n), then

$$\partial_{v_n} \tilde{\Phi}_n = \frac{c_n}{|J_n|} \left(\frac{1}{2} \Phi_{n2} - \frac{1}{2} \Phi_{n1} \right) = 0$$

Taking (2.1) into account, we can write the equations of the above interpolation problem in the form

$$\frac{c_n}{2|J_n|} (\Phi_{n2} - \Phi_{n1}) = \Psi_{n0} a_{nm}, \quad \forall n, m \tag{2.2}$$

The algorithm for solving a system of equations of the form (2.2) for Φ_{nk} ($k = 1, 2$) uses the conditions for consistency between the elements $\Phi_{n1} = \Phi_{(n-1)2}$, $\Phi_{n2} = \Phi_{(n+1)1}$ (hence the number of required nodal values Φ_{nk} is equal to the number of values Ψ_{n0} known from the solution of (1.11)) and the following condition for global interpolation (at the points of S_{Δ}) of the normal derivative (see (2.1))

$$\sum_{n=1}^N \int_{\Delta s_n} \frac{c_n}{|J_n|} \sum_{k=1}^2 \Phi_{nk} \partial_{\eta} \Psi_k ds_n(y) = 0 \tag{2.3}$$

which corresponds (see below) to the condition for the normal derivative of the harmonic function.

Taking (2.2) into account, we can write (1.11) in the form

$$\sum_{n=1}^N \frac{c_n}{2|J_n|} (\Phi_{n2} - \Phi_{n1}) = \sum_{n=1}^N f_n, \quad \forall m \tag{2.4}$$

In Section 3 it will be established that system (2.4) is equivalent to the system of DBEs for the ‘‘Galerkin’’ weak variational formulation of the BEM.

Remark. The form of the interpolation (2.1) in the global coordinates can be established according to (2.3)

$$\int_{\Delta s_n} \partial_{v_n} \tilde{\Phi}_n ds(y_n) = \sum_{k=1}^2 \Phi_{nk} \int_{\Delta s_n} \sum_{i=1}^2 \partial_{y_n^{(i)}} \Psi_k \cos \alpha_{in} ds(y_n)$$

Here the integrals for $i = 1, 2$ can be computed uniquely. For example, for $i = 1$ we have

$$\begin{aligned} \int_{\Delta s_n} \partial_{y_n^{(1)}} \Psi_k \cos \alpha_{1n} ds(y_n) &= \int_{Y_n^{(1)}}^{Y_n^{(2)}} \partial_{y_n^{(1)}} \Psi_k \cos \alpha_{1n} \left(-\frac{dy_n^{(1)}}{\cos \alpha_{2n}} \right) = \\ &= -\frac{\cos \alpha_{1n}}{\cos \alpha_{2n}} [\Psi_k(y_n^{(1)} = Y_n^{(2)}) - \Psi_k(y_n^{(1)} = Y_n^{(1)})] \end{aligned}$$

Next we use (1.8) and ψ_k at the nodes $k = 1, 2$

$$\begin{aligned} \psi_1(Y_{n2}^{(1)}) &= \psi_1(\eta = 1) = 0; & \psi_1(Y_{n1}^{(1)}) &= \psi_1(\eta = -1) = 1 \\ \psi_2(Y_{n2}^{(1)}) &= \psi_2(\eta = 1) = 1; & \psi_2(Y_{n1}^{(1)}) &= \psi_2(\eta = -1) = 0 \end{aligned}$$

We obtain

$$\sum_{k=1}^2 \Phi_{nk} \int_{\Delta s_n} \partial_{y_n^{(i)}} \psi_k \cos \alpha_{1n} ds(y_n) = \frac{\bar{Y}_n^{(2)}}{\bar{Y}_n^{(1)}} (\Phi_{n2} - \Phi_{n1}), \quad \forall n$$

Similar calculations for $k = 2$ finally yield

$$\int_{\Delta s_n} \partial_{v_n} \tilde{\varphi}_n ds_n = c_n (\Phi_{n2} - \Phi_{n1}), \quad \forall n$$

The same result can also be obtained by integrating (2.1) (making use of the equality $ds(y_n) = |J_n| d\eta$)

$$\int_{\Delta s_n} \partial_{v_n} \tilde{\varphi}_n ds(y_n) = \frac{c_n}{2|J_n|} (\Phi_{n2} - \Phi_{n1}) \int_{-1}^{+1} |J_n| d\eta = c_n (\Phi_{n2} - \Phi_{n1}), \quad \forall n$$

3. We will show that the density of the approximating SLP (1.3) can be constructed using a numerical Galerkin-type process employing the interpolation equations (2.2).

We recall that Galerkin’s method uses the orthogonality of the error in satisfying the differential equation of the boundary-value problem to the set of coordinate functions satisfying all boundary conditions of the problem [14]. There is a version [14] of the method for the case of natural boundary conditions and also various extensions of the Galerkin process [14] using the aforementioned idea of the method for a different choice of the system of coordinate functions: Petrov’s extension the method of separating the domain, and the collocation method.

Here, to solve problem (1.1) approximately as an approximating SLP (1.3), we propose a process of finding the density which uses the condition for the error in satisfying the boundary condition of the problem to be orthogonal to the basis functions of the BEM and a discrete analogue of the condition for the normal derivative of a harmonic function. The process is justified numerically, since the resulting system of DBEs is equivalent to system (2.4), which follows from the system of discrete BIEs, the convergence of the sequence of approximating SLPs having been established in [6].

We require that the condition

$$\int_{\Delta s_n} (\partial_{v_n} \tilde{\varphi}_n - f_n) \sum_{l=1}^2 \psi_l ds(y_n) = 0, \quad \forall n \tag{3.1}$$

should be satisfied at the points of the elements Δs_n , where $\partial_{v_n} \tilde{\varphi}_n$ is defined by (2.1) and f_n by (1.10).

Changing to local coordinates, we rewrite (3.1) in the form

$$\frac{c_n}{|J_n|} \int_{-1}^{+1} \left(\sum_{k=1}^2 \Phi_{nk} \partial_{\eta} \psi_k \right) \sum_{l=1}^2 \psi_l |J_n| d\eta = \frac{1}{|J_n|} \int_{-1}^{+1} \left(\sum_{i=1}^2 y_n^{(i)} \bar{Y}_n^{(i)} \right) \sum_{l=1}^2 \psi_l |J_n| d\eta, \quad \forall n \tag{3.2}$$

The right-hand side is written for the general case of geometric interpolation, when $y_n^{(i)}$ ($i = 1, 2$) is defined by (1.4), and (1.8) is used. The transformation of the left-hand side of (3.2) involves evaluating the integrals $\int \partial_{\eta} \psi_k \psi_l d\eta$ over the interval $[-1, 1]$, while the right-hand side involves $\int \psi_l d\eta$ and $\int \eta \psi_l d\eta$.

As a result, (3.2) becomes

$$c_n (\Phi_{n2} - \Phi_{n1}) = 2 \sum_{i=1}^2 \bar{Y}_n^{(i)} \bar{Y}_n^{(i)}, \quad \forall n$$

which, after summing over $n = 1, \dots, N$, is equivalent to (2.4). For the middle node $k' = 0$ of the element Δs_n we have $\bar{Y}_n^{(i)} = Y_{n0}^{(i)}$.

Establishing the equivalence of the systems of DBEs for the two formulations of the BEM considered above involves condition (2.3), which is a discrete analogue of the condition for the normal derivative of a harmonic function. Indeed, the normal derivative of the approximating SLP $\varphi_N(x)$ (see (1.3)) for $x \in S_{\Delta}$ corresponds to the left-hand side of the discrete BIE (see (1.7)). Since the approximating SLP

is a harmonic function in G_Δ , the integral of the normal derivative over S_Δ must be zero. This is so because the integral of the right-hand side of the discrete BIE over S_Δ is zero, which corresponds to the solubility condition for the Neumann problem [14] and is easily verified for the approximation (1.10). The change from the system of discrete BIEs of the form (1.7) to the system (2.4) using (2.2) is justified if the global interpolation of the normal derivative conforms to condition (2.3). Thus, in the realization of the weak formulation algorithm Galerkin's equations are supplemented by condition (2.3).

It is also natural to try to establish a correspondence between the extension of the Galerkin process presented in Section 3 and Galerkin's method for the direct solution of the BIE of Fredholm type [14] for the basis functions of the BEM. Without touching upon the issue of the convergence of the "Galerkin" boundary element approximations for the solution of the BIE, one can assert that if the discrete BIE holds for each functional node (see (1.7) for constant boundary elements), then multiplying each BIE by $\Sigma\psi_l$ ($l = 1, 2$), integrating the result over $S_{\Delta n}$, and taking the sum over $n = 1, \dots, N$, we obtain a realization of the Galerkin process for the solution of the BIE.

The approximate solution of problem (1.1) both in the formulation based on the BIE and in the "Galerkin" formulation can be represented [7] as a linear combination of products of the nodal values Ψ_{nk} of the SLP density (1.3) and the integral "influence" functions of the k 'th mode and n th boundary element, which can be computed using (1.5). Each term in this linear combination is a harmonic function in the domain G_Δ with boundary S_Δ . Then, for such functions the approximating problem for (1.1) is equivalent (see, for example, [7]) to the variational problem for the corresponding BF. It can be verified that the boundary-element approximations constructed above (for the nodal values Ψ_{nk} found from the formulation based on the BIE or the "Galerkin" formulation) satisfy the variational equation for the BF and, consequently, they minimize the functional. In [6] it was established that the minimizing sequence for the BF converges to the solution of problem (1.1) as $N \rightarrow \infty$.

We have therefore established a relationship between the formulation of the BEM based on a SLP and the "Ritz" formulation.

The above argument can also be used to relate the formulations of the BEM for solving problems in the linear theory of elasticity.

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